Quantum Machine Learning Exercise Sheet

August 18, 2022

Exercises

1.) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- **a.**) What is the singular value decomposition of A?
- **b.**) Give a singular value decomposition of B!
- **c.**) What is the eigenvalue decomposition of B?
- **d.**) Give a singular value decomposition of B where the right singular vectors coincide with the eigenvectors!

2.) Let $A = U^\dagger \Sigma V = \sum_{i=1}^m \sigma_i |u_i\rangle\langle v_i|$, where u_i, v_i are the columns of the unitaries U, V and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$ are the singular values of A. Prove that $A^\dagger A = \sum_{i=1}^m \sigma_i^2 |v_i\rangle\langle v_i|$.

3.) We know that every Hermitian matrix $H \in \mathbb{C}^{n\times n}$ has a decomposition of the from $\sum_{i=1}^n \lambda_i |\psi_i\rangle\langle\psi_i|$, where $\lambda_i \in \mathbb{R}$ and $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. Prove that every $A \in \mathbb{C}^{m\times n}$ has a singular value decomposition.

4.) Let $R\colon |0\rangle|i\rangle \mapsto \frac{|A_{i.}\rangle|i\rangle}{\|A_{i.}\|}$ and $C\colon |0\rangle|j\rangle \mapsto \frac{|j\rangle|a\rangle}{\|a\|}$, where $|a\rangle = \sum_{i=1}^m a_i|i\rangle$ is the vector of row norms of A such that $a_i = \|A_{i.}\|$. Show that $U = R^\dagger C$ is a block-encoding of $A/\|A\|_F$.

5.) Show that $U^\dagger(2|0\rangle\langle 0| \otimes I_2 - I_{12})U$ is a block-encoding of $2\frac{A^\dagger A}{\|A\|_F^2} - I$ if U is a block-encoding of $A/\|A\|_F$:

$$U = \begin{pmatrix} A/\|A\|_F & \cdot \\ \cdot & \cdot \end{pmatrix}.$$

6.) (Quantum rejection sampling) Suppose you have a quantum state $\sum_{i=1}^m \sqrt{p_i}|i\rangle$ and we have a distribution q such that $q \leq c \cdot p$. Show that you can prepare the state $\sum_{i=1}^m \sqrt{q_i}|i\rangle$ with $\mathcal{O}(\sqrt{c})$ queries for an oracle $O\colon |i\rangle|0\rangle \mapsto |i\rangle|q_i/p_i\rangle$ that outputs the ratio $\frac{q_i}{p_i}$ as a fixed point binary number.

7.) (SWAP test) Suppose you get quantum states from a source such that either

- every time you get the same (unknown) quantum state $|\psi\rangle$
- every time you get $|i\rangle$ for a uniformly random $i \in 0, 1, \ldots, n-1$.

Request two quantum states from the source and append an ancilla qubit in the $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ state. Controlled by the ancilla qubit SWAP the two copies. Measure the ancilla qubit in the $|\pm\rangle$ basis. What is the probability of the + outcome in the above two scenarios?

8.) [dW19, Chapter 18 Exercise 7]

References

[dW19] Ronald de Wolf. Quantum computing: Lecture notes (version 4), 2019. arXiv: `1907.09415v4`