2025 Quantum Computing Homework Nr. 8

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Homework exercises – you can earn 10 points in total! The first two exercises are worth 2 + 3 points respectively, while the last is worth 5 = 1+2+2. On the last page you can find some hints where indicated by (**H**).

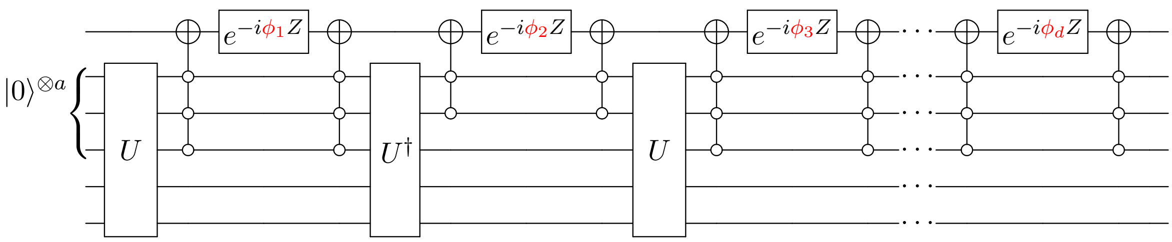
Reminder

Let $A = \sum_{i=1}^{d} \sigma_i |w_i\rangle\langle v_i|$ be a singular value decomposition of A, i.e., $v_i : i \in \{1, 2, \ldots, d\}$ are orthonormal vectors as well as $w_i : i \in \{1, 2, \ldots, d\}$, and the singular values $0 \leq \sigma_i : i \in \{1, 2, \ldots, d\}$ are ordered decreasingly.

Theorem 1 (Quantum Singular Value Transformation (QSVT)[GSLW19]). *Let $P\colon [-1, 1] \mapsto [-1, 1]$ be a degree-d odd polynomial map. Suppose that U is a block-encoding of $A = (\langle 0^a| \otimes I)U(|0^b\rangle \otimes I)$. Then $V := (H \otimes I)U_\Phi(H \otimes I)$ is a block-encoding of*

$$\sum_{i=1}^{d} P(\sigma_i)|w_i\rangle\langle v_i| = (\langle 0^{a+1}| \otimes I)V(|0^{b+1}\rangle \otimes I), \tag{1}$$

*where $\Phi \in \mathbb{R}^d$ is efficiently computable from P and U_Φ is the following circuit:**



Exercises

1.) Show that the operator norm of A is $\|A\| = \sigma_1$.

2.) Let $f\colon \mathbb{R} \to \mathbb{C}$, and $P \in \mathbb{C}[x]$ a polynomial such that $|f(x) - P(x)| \leq \varepsilon$ for all $x \in S \subseteq \mathbb{R}$. Suppose that the singular values of A are elements of the set $\sigma_i \in S$ for all $i \in \{1, 2, \ldots, d\}$. Show that $B := \sum_{i=1}^{d} f(\sigma_i)|w_i\rangle\langle v_i|$ and $\widetilde{B} := \sum_{i=1}^{d} P(\sigma_i)|w_i\rangle\langle v_i|$ are ε-close, i.e., $\left\|B - \widetilde{B}\right\| \leq \varepsilon$.

3.) This exercise shows how to solve a linear equation of the form $Ax = b$, when the equation might be under/over-determined. The least square solution is given by A^+b, where A^+ is the Moore-Penrose pseudoinverse of A defined as $A^+ := \sum_{i\colon \sigma_i \neq 0} \frac{1}{\sigma_i}|v_i\rangle\langle w_i|$.

(a) Suppose that U is a block-encoding of A, s.t., $A = (\langle 0| \otimes I)U(|0\rangle \otimes I)$, and $\|A^+\| \leq \kappa$. Describe a bounded set S that contains the singular values of any such matrix A, but is disjoint from some interval of the form $(0, a)$ for some $a > 0$. What is the largest a that we can choose?

*The empty dots denote control on the state $|0\rangle$. The generalized $CNOT$/Toffoli gates are controlled by $|0^a\rangle$ and $|0^b\rangle$ on the right- and left-hand sides of U respectively in the circuit – in this example circuit $a = 3$, $b = 2$.

(b) Construct an approximate block-encoding V of $A^+/(2\kappa)$ such that

$$\left\| A^+/(2\kappa) - (\langle 00| \otimes I) V (|00\rangle \otimes I) \right\| \leq \varepsilon,$$

and V uses $\mathcal{O}(\kappa \log(1/\varepsilon))$-times the block-encoding U and its inverse $U^\dagger$. (**H**)

(c) Assume for simplicity that $A^+/(2\kappa) = (\langle 00| \otimes I) V (|00\rangle \otimes I)$ holds exactly. Suppose that W is a quantum circuit that maps $|0^n\rangle \to |b\rangle$. Give a quantum algorithm that prepares a quantum state proportional to $A^+|b\rangle$ with high probability with $\mathcal{O}(\kappa)$ uses of the quantum circuits V and W. What does it tell us about the complexity of preparing a quantum state proportional to the least-square solution $A^+|b\rangle$ given the block-encoding of A?

Hints

Exercise 3b: You can use the following polynomial approximation result [Gil19, Corollary 3.4.13.] without proof: for every $\kappa > 1$ and $\varepsilon < 1$ there is an odd polynomial $P \in \mathbb{R}[x]$ of degree $\mathcal{O}(\kappa \log(1/\varepsilon))$, such that $|P(x)| \leq 1$ for all $x \in [-1, 1]$, and $|P(x) - 1/(2\kappa x)| \leq \varepsilon$ for all $x \in [1/\kappa, 1]$.

References

[Gil19] András Gilyén. *Quantum Singular Value Transformation & Its Algorithmic Applications.* PhD thesis, University of Amsterdam, 2019.

[GSLW19] András Gilyén, Yuan Su, Guang Hao Low, and Nathan Wiebe. Quantum singular value transformation and beyond: Exponential improvements for quantum matrix arithmetics. In *Proceedings of the 51st ACM Symposium on the Theory of Computing (STOC)*, pages 193–204, 2019. arXiv: `1806.01838`