

2025 Quantum Computing Homework Nr. 7

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Homework exercises – you can earn 10(+2) points in total! The second exercise is worth 2(+2) points while the others 4. On the last page you can find some hints where indicated by **(H)**.

Reminder

Recall the *Spectral Theorem (Főttengely Tétel in Hungarian)*: for every self-adjoint matrix $A \in \mathbb{C}^{d \times d}$ there exists a unitary $V \in \mathbb{C}^{d \times d}$ that diagonalizes A , i.e., $A = V \cdot D \cdot V^\dagger$ where $D \in \mathbb{R}^{d \times d}$ is a diagonal matrix containing the eigenvalues of A , and the columns of V are the corresponding eigenvectors. Let $|v_i\rangle$ be the i -th column of V and let λ_i be the i -th diagonal entry of D . In Dirac notation we can write this eigenvalue decomposition as $A = \sum_{i=0}^{d-1} \lambda_i |v_i\rangle\langle v_i|$.

Exercises

- 1.) A generalization of the Spectral Theorem for arbitrary matrices is called *Singular Value Decomposition*. This exercise walks you through a proof of this generalization by a reduction to the above spectral theorem. Let $A \in \mathbb{C}^{k \times d}$.
 - (a) Show that $A^\dagger A$ is self-adjoint and all of its eigenvalues are ≥ 0 , so there is an eigenvalue decomposition $A^\dagger A = \sum_{i=0}^{d-1} \sigma_i^2 |v_i\rangle\langle v_i|$, where $\sigma_i \geq 0$ and the vectors $|v_i\rangle$ are orthonormal.
 - (b) Show that $\{|w_i\rangle := A|v_i\rangle/\sigma_i : \sigma_i > 0\}$ form an orthonormal system of vectors.
 - (c) Show that $A = \sum_{i: \sigma_i \neq 0} \sigma_i |w_i\rangle\langle v_i|$.
 - (d) Show that there exists unitaries $V \in \mathbb{C}^{d \times d}$, $W \in \mathbb{C}^{k \times k}$ and a diagonal matrix $\Sigma \in \mathbb{R}_{\geq 0}^{k \times d}$ containing the numbers σ_i on the diagonal such that $A = W \cdot \Sigma \cdot V^\dagger$.

The decomposition in Exercise 1c-1d is called Singular Value Decomposition. The numbers σ_i are called the singular values of A (usually ordered in decreasing order for convenience) and the vectors w_i and v_i are called the corresponding left and right singular vectors of A respectively.

- 2.) Linear combination of unitaries: Suppose that $A_0 = (|0^a\rangle \otimes I)U_0(|0^b\rangle \otimes I)$ and $A_1 = (|0^a\rangle \otimes I)U_1(|0^b\rangle \otimes I)$. Given $\alpha_0, \alpha_1 \in \mathbb{R}$ such that $|\alpha_0| + |\alpha_1| = 1$, construct a circuit that uses $U = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$ and two single qubit gates to implement a block-encoding V of $\alpha_0 A_0 + \alpha_1 A_1$ such that

$$(|0^{a+1}\rangle \otimes I)V(|0^{b+1}\rangle \otimes I) = \alpha_0 A_0 + \alpha_1 A_1. \quad \text{(H)}$$

★ For 2 bonus points: construct a similar circuit in the general case $\alpha_0, \alpha_1 \in \mathbb{C}$.

- 3.) From Ronald's lecture notes [dW19, Chapter 9 Exercise 10]: Block-encoding an s -sparse Hermitian matrix A with $\|A\| \leq 1$ (see Section 9.4). Assume for simplicity that the entries of A are real.
 - (a) Show how to implement W_1 using an $O_{A,loc}$ -query and a few other A -independent gates. For simplicity you may assume s is a power of 2 here, and you can use arbitrary single-qubit gates, possibly controlled by another qubit.
(Note that the same method allows to implement W_3 .)

- (b) Show how to implement W_2 using an O_A -query, an O_A^{-1} -query, and a few other A -independent gates (you may use auxiliary qubits as long as those start and end in $|0\rangle$). Note that W_2 just implements a rotation on the first qubit, by an angle that depends on A_{kj} . There's no need to write out circuits fully down to the gate-level here; it suffices if you describe the idea precisely.
- (c) Show that the $(0^{n+1}i, 0^{n+1}j)$ -entry of $W_3^{-1}W_1$ is $1/s$ if $A_{ij} \neq 0$, and is 0 if $A_{ij} = 0$.
- (d) Show that the $(0^{n+1}i, 0^{n+1}j)$ -entry of $W_3^{-1}W_2W_1$ is exactly A_{ij}/s .

Hints

Exercise 2: Remember from the lecture that $(\langle 0^{a+1}| \otimes I)(H \otimes I)U(H \otimes I)(|0^{b+1}\rangle \otimes I) = \frac{1}{2}(A_0 + A_1)$. Replace the Hadamrd gates with appropriate single-qubit gates.

References

[dW19] Ronald de Wolf. Quantum computing: Lecture notes, 2019. arXiv: 1907.09415v5