2025 Quantum Computing Homework Nr. 10

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Homework exercises – you can earn 14 points in total! Each exercise is worth one more point than its number. On the last page you can find some hints where indicated by (**H**).

Background

Let $A = \sum_{i=1}^{d} \sigma_i |w_i\rangle\langle v_i|$ be a singular value decomposition of A, i.e., $v_i \colon i \in \{1, 2, \ldots, d\}$ are orthonormal vectors as well as $w_i \colon i \in \{1, 2, \ldots, d\}$, and the singular values $0 \leq \sigma_i \colon i \in \{1, 2, \ldots, d\}$ are ordered decreasingly. The Schatten-p norm $\|A\|_p$ of the matrix A is defined as the ℓ_p norm of its singular values $\sqrt[p]{\sum_i \sigma_i^p}$, for any $p \in [1, \infty]$.

Hölder's inequality states that $\|AB\|_1 \leq \|A\|_1 \|A\|_\infty$, while the trace-norm inequality states that $|\mathrm{Tr}[A]| \leq \|A\|_1$.

The Schatten-2 norm is also known as the Frobenius norm, and it is the norm induced by the Hilbert-Schmidt inner product of matrices $\langle A, B\rangle_{HS} = \mathrm{Tr}\big[A^\dagger B\big]$, $\|A\|_2 = \sqrt{\langle A, A\rangle_{HS}}$.

Exercises

1.) Let us define a superoperator $\mathcal{S}[\cdot]$ through its action as $\mathcal{S}[\rho] = A\rho B^\dagger$. Show that the adjoint of S (with respect to the Hilbert-Schmidt inner product) acts as $\mathcal{S}^\dagger[\rho] = A^\dagger \rho B$. (**H**)

2.) Let us define a detection event D as a collection of some outcomes of a POVM. Show that the difference in the probability of any detection event is bounded by the trace norm of the difference of the corresponding density operators, i.e., (**H**)

$$|\mathbb{P}_\rho[D] - \mathbb{P}_\sigma[D]| \leq \frac{1}{2}\|\rho - \sigma\|_1.$$

For this reason $\frac{1}{2}\|\rho - \sigma\|_1$ is called the trace distance between ρ and σ, and it is a frequently used distance measure on density operators.

3.) Show that the transpose superoperator $\mathcal{T}[M] = M^T$ is positive, but not completely positive, i.e., there exists $\rho \succeq 0$ such that $(\mathcal{I} \otimes \mathcal{T})[\rho] \not\succeq 0$. (**H**)

4.) Let's use some quantum magic to detect working / inoperational mines using a quantum cat! If the cat steps ($|{\downarrow}\rangle$) on a working mine, then it kicks the bucket ("feldobja a talpát"): $|$$\rangle\langle$$|$, but if it steps on a dud (non-working mine), then nothing happens. I.e., we can model the situation with a unitary that is the identity in case of a dud but otherwise acts as follows:

$U = |{\uparrow}\rangle\langle{\uparrow}| \otimes I + |{\downarrow}\rangle\langle{\downarrow}| \otimes |$$\rangle\langle$$| + \ldots,$

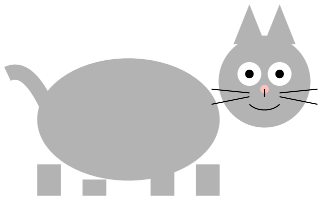
where $\ldots$ hides terms that act on the subspace spanned by $|{\downarrow}\rangle \otimes |\psi\rangle$ such that $\langle\psi|$$\rangle = 0$.

Devise a protocol with the following properties: (**H**)

(a) If the mine was a dud, then we correctly report it being a dud.

(b) If the mine was working, then we correctly detect that it is working and the mine explodes with probability at most ε.

(c) We only use the quantum cat for testing purposes at most $\mathcal{O}(1/\varepsilon)$ times.



Hints

Exercise 1: Use the cyclicity of trace: $\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB]$.

Exercise 2: Use Hölder's inequality.

Exercise 3: Consider some nice entangled input state.

Exercise 4: Use the quantum Zeno effect, with the cat facilitating the measurements!