2023 Quantum Computing Homework Nr. 7

András Gilyén

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Homework exercises – fall break edition – you can earn 14 points in total as opposed to the usual 10 points! Each exercise is worth one more point than its number. On the last page you can find some hints where indicated by (**H**). Some notation: I_n denotes the identity operator on n-qubits, and $\mathbf{1}_S$ denotes the indicator of the set S, i.e., $\mathbf{1}_S(x)$ is 1 if $x \in S$ and 0 otherwise.

Exercises

1.) From Ronald's lecture notes [dW19, Chapter 7 Exercise 1]:

(a) Suppose $n = 2$, and $x = x_{00}x_{01}x_{10}x_{11} = 0001$. Give the initial, intermediate, and final superpositions in Grover's algorithm, for $k = 1$ queries. What is the success probability?

(b) Give the final superposition for the above x after $k = 2$ iterations. What is now the success probability?

2.) Suppose A is a quantum circuit that prepares some n-qubit state $|\psi\rangle$, i.e., $A\colon |0^n\rangle \mapsto |\psi\rangle = \sqrt{1-p}|0\rangle|B\rangle + \sqrt{p}|1\rangle|G\rangle$, where $|B\rangle$ and $|G\rangle$ are some $(n-1)$-qubit pure states and p is known.

(a) Give an algorithm that outputs the state $|G\rangle$ with high success probability and uses A and $A^\dagger$ a total of $\mathcal{O}\left(\frac{1}{\sqrt{p}}\right)$ times. Analyse the algorithm and prove the running time bound.

(b) Give an algorithm that outputs the state $|G\rangle$ with certainty and still uses A and $A^\dagger$ a total of $\mathcal{O}\left(\frac{1}{\sqrt{p}}\right)$ times. (**H**)

(c) Suppose in the unstructured search problem we know the number of marked elements t. Give a quantum algorithm that makes $\mathcal{O}\left(\sqrt{\frac{N}{t}}\right)$ queries and outputs a marked element with certainty.

3.) Let P be a symmetric Markov chain on the discrete state space $V = \{0, 1, \ldots, 2^n - 1\}$. Let $M \subseteq V$ be the set of marked elements. Suppose we have the following walk operators

$$\text{Update: } U \in \mathbb{C}^{2^{2n+2}\times 2^{2n+2}} \text{ mapping } |0\rangle|0^n\rangle|0\rangle|v\rangle \mapsto \sum_{w\in V}\sqrt{P_{w,v}}|0\rangle|w\rangle|0\rangle|v\rangle$$

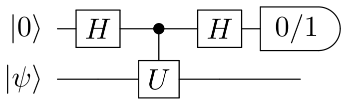
$$\text{Check: } C \in \mathbb{C}^{2^{n+1}\times 2^{n+1}} \text{ mapping } |0\rangle|v\rangle \mapsto |\mathbf{1}_M(v)\rangle|v\rangle$$

(a) Let $W' := U^\dagger(I_{n+1}\otimes C^\dagger)\mathrm{SWAP}(I_{n+1}\otimes C)U$, where SWAP swaps the first and last $(n+1)$-qubit registers. Prove that $(\langle 0^{n+2}|\otimes I_n)W'(|0^{n+2}\rangle\otimes I_n) = P_U$, where P_U is the matrix we get after zeroing out the columns and rows of P corresponding to marked elements, i.e., show that $\langle 0^{n+2}|\langle w|W'|0^{n+2}\rangle|v\rangle = (P_U)_{w,v}$ for every $w, v \in V$.

(b) Let $W := (2|0^{n+2}\rangle\langle 0^{n+2}|\otimes I_n - I_{2n+2})W'$. Prove that $(\langle 0^{n+2}|\otimes I_n)W^t(|0^{n+2}\rangle\otimes I_n) = T_t(P_U)$, where $T_t(x) = \cos(t\arccos(x))$ is the t-th Chebyshev polynomial of the first kind. (**H**)

4.) Let $P_U = \sum_i \lambda_i |e_i\rangle\langle e_i|$ be an orthogonal eigendecomposition of P_U from Exercise 3. In the lecture we have proven that $|u\rangle = \sqrt{\frac{1}{|V|}}\sum_{v\in V}|v\rangle = c_i|e_i\rangle$ has the property that $\sum_{i:\, \lambda_i \geq 1-\frac{1}{6HT}} c_i^2 \leq \frac{1}{6}$. Our goal is to distinguish the cases $M = \emptyset$ from $M \neq \emptyset$ under the promise that if $M \neq \emptyset$ then the hitting time $HT \leq R$ for some known $R \in \mathbb{N}$.

(a) What is the probability of getting measurement outcome $|0\rangle$ after executing the following quantum circuit known as the *Hadamard-test* and measuring the first qubit?



(b) Suppose A is a unitary such that $(\langle 0^{n+2}|\otimes I_n)A(|0^{n+2}\rangle\otimes I_n) = (P_U)^{12R+1}$. Give an algorithm that uses a controlled-A gate once, and with probability at least $\frac{1}{3}$ detects the presence of marked elements. More precisely, if $M \neq \emptyset$ it outputs 1 with probability at least $\frac{1}{3}$, but if $M = \emptyset$ it is only allowed to output 0. (**H**)

(c) An intriguing property of Chebyshev polynomials is that [**?**]

$$x^t = \sum_{i=0}^{t} 2^{-t}\binom{t}{i} T_{|2i-t|}(x).$$

Based on this observation give an algorithm that detects $M \neq \emptyset$ with probability at least $\frac{1}{3}$ and uses an expected number of $\mathcal{O}\left(\sqrt{R}\right)$ quantum walk steps W from Exercise 3b. (**H**)

(d) Give an algorithm as above in part 4c, which still detects $M \neq \emptyset$ with probability at least $\frac{1}{6}$ but uses at most $\mathcal{O}\left(\sqrt{R}\right)$ quantum walk steps W. Finally, boost the success probability to $\frac{2}{3}$.

Hints

Exercise 2b: Can you think of some values of p for which you could solve the problem exactly? If so how does that help you to solve the general case exactly?

Exercise 3b: Use the recurrence relation $T_0(x) = 1$, $T_1(x) = x$, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, and the fact that W' is self-inverse.

Exercise 4b: Use the Hadamard test. Observe that for all $x \in (0, 1]$: $(1-x)^{\frac{2}{x}} \leq \frac{1}{6}$ and thereby we have that $\lambda_i < 1 - \frac{1}{6HT} \Rightarrow \lambda_i^{(12R+1)} \leq \frac{1}{6}$. Also remember that $P|u\rangle = |u\rangle$.

Exercise 4c: Use the Hadamard test on a t-step quantum walk, where the number t is chosen in a random fashion. Also you may use the fact that the first central absolute moment of the binomial distribution $B(n, \frac{1}{2})$ is $\mathcal{O}(\sqrt{n})$ [?].

References

[dW19] Ronald de Wolf. Quantum computing: Lecture notes, 2019. arXiv: `1907.09415v5`