Quantum Fourier transform beyond Shor's algorithm: Exercise Sheet 3

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Feel free to skip exercises that you find too easy or hard. On the last page you can find some hints where indicated by (**H**).

Exercises

1.) Prove that for the continuous Fourier transform $\mathcal{F}$ shifting in one domain corresponds to point-wise phase multiplication in the other (Fourier transformed) domain.

2.) (**H**) Give a succinct formula for $H^{\otimes d}|j\rangle$. Based on your formula prove the correctness of the Bernstein-Vazirani algorithm!

3.) (**H**) Suppose you have access to a function $f\colon \mathbb{Z}_N^d \to \mathbb{Z}_N$ via a binary oracle

$$O_f\colon |x\rangle|b\rangle \mapsto |x\rangle|b+f(x) \mod N\rangle.$$

Use phase kickback to construct a phase oracle

$$U_f\colon |x\rangle \mapsto e^{\frac{2\pi i}{N}f(x)}|x\rangle,$$

with a single query to O_f.

4.) (**H**) Suppose we have a random variable X taking its values from $\{0,1,\ldots,d-1\}$. The probability distribution of X is represented by the quantum state

$$|\psi\rangle = \sum_{i=0}^{d-1}\sqrt{p_i}|i\rangle.$$

Let $f\colon \{0,1,\ldots,d-1\} \to [0,1]$ be a function which is represented by a linear operator F acting as

$$F\colon |i\rangle|0\rangle \mapsto |i\rangle\Big(\sqrt{1-f(i)}|0\rangle + \sqrt{f(i)}|1\rangle\Big).$$

(a) How can you get an ε-precise estimate of $\mathbb{E}[f(X)]$ using $O(\frac{1}{\varepsilon})$ applications of F?

(b) Give an efficient quantum algorithm estimating the expectation of the random variable X.

(c) Give an efficient quantum algorithm estimating the variance of the random variable X.

5.) (a) Given state preparation unitaries $U\colon |0\rangle|0\rangle^{\otimes a} \mapsto |0\rangle|\psi\rangle + |1\rangle|\tilde{\psi}\rangle$ and $V\colon |0\rangle|0\rangle^{\otimes a} \mapsto |0\rangle|\phi\rangle + |1\rangle|\tilde{\phi}\rangle$ construct a unitary W such that $\langle\bar{0}|W|\bar{0}\rangle = \langle\psi|\phi\rangle$

(b) Given state preparation unitaries $U := \sum_x U_x \otimes |x\rangle\langle x|$ and $V := \sum_x V_x \otimes |x\rangle\langle x|$ controlled by the second register, where $U_x\colon |0\rangle|0\rangle^{\otimes a} \mapsto |0\rangle|\psi_x\rangle + |1\rangle|\tilde{\psi}_x\rangle$ and $V_x\colon |0\rangle|0\rangle^{\otimes a} \mapsto |0\rangle|\phi_x\rangle + |1\rangle|\tilde{\phi}_x\rangle$ are $(a+1)$-qubit state-preparation unitaries for some (subnormalized) a-qubit quantum states $|\psi_x\rangle, |\phi_x\rangle$, construct a block-encoding of the diagonal matrix $\mathrm{diag}(\langle\psi_x|\phi_x\rangle)$

(c) (**H**) Suppose for simplicity that $|\psi\rangle \in \mathbb{R}^d$, and $U\colon |\bar{0}\rangle \to |\psi\rangle$ is a state preparation unitary. Use Jordan's algorithm to produce a coordinate-wise ε-precise estimation of $|\psi\rangle$ with $\widetilde{\mathcal{O}}\Big(\sqrt{d}/\varepsilon\Big)$ uses of U (and its inverse). With how many uses of U can you give an ε-precise estimation with respect to the Euclidian distance?

6.) Suppose you have access to a function $f\colon [0,1]^d \to \mathbb{R}$ satisfying

$$|f(x) - f(0) - g \cdot x| \leq \frac{1}{42N}$$

for at least 99.9% of the points in G^d, where each coordinate of $g \in \mathbb{R}^d$ is at most $1/3$ in absolute value. Show that for every individual coordinate it holds, that with high probability ($\geq 2/3$) Jordan's algorithm outputs a $1/N$ precise estimate. How can you boost this success probability such that the output gives $1/N$-precise coordinate-wise estimator with probability at least $1-\delta$?

Hints

Exercise 2: Use that the Hadamard gate is self-inverse.

Exercise 3: Recall, that in the Bernstein-Vazirani ($\mathbb{Z}_2$ case) the second column of its Fourier transform is used as an ancilla state for implementing the phase via kick back.

Exercise 4:

1. Calculate $F|\psi\rangle$ and apply amplitude estimation.
2. Choose $f(i) = \frac{i}{d-1}$.
3. Choose $f(i) = \frac{i^2}{(d-1)^2}$, and use the previous part.

Exercise 5.c: Construct a block-encoding of the linear function $f\colon \vec{x} \to \langle \vec{x}|\psi\rangle/\sqrt{d}$ using the construction from part b. Note that we only require query efficiency, not gate efficiency!