Quantum Fourier transform beyond Shor's algorithm: Exercise Sheet 2

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Feel free to skip exercises that you find too easy or hard. On the last page you can find some hints where indicated by (**H**).

Exercises

1.) What happens if you perform phase estimation on a superposition $\sum_j \alpha_j|\psi_j\rangle$ of orthonormal eigenvectors of U with eigenphases φ_j?

2.) (a) Determine the eigenvalues of the matrix

$$G = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

(b) Recall Grover's search problem: We are given phase oracle access to the function $f\colon \{0,1,\ldots,2^n-1\} \to \{0,1\}$:

$$O_f\colon |x\rangle \mapsto (-1)^{f(x)}|x\rangle.$$

Our goal is to find a solution x that satisfies $f(x) = 1$. Grover's algorithm uses the Grover operator $H^{\otimes n}(2|0^n\rangle\langle 0^n| - I_n)H^{\otimes n}O_f$ to solve this.

What do we get if we apply phase estimation to the Grover operator and initial state $H^{\otimes n}|0^n\rangle$?

(c) How can we use this to conclude about the number of solutions of the search problem?

3.) (**H**) Show that phase estimation cannot be made even "approximately" deterministic in general. More precisely prove that it is impossible to give a black-box phase estimation circuit which for arbitrary input U and eigenstate $|\psi\rangle$ with corresponding eigenvalue $e^{2\pi i\varphi}$ has a most probable estimate $\tilde{\varphi}$ outputted with probability larger than $1/2$ that also satisfies $|\varphi - \tilde{\varphi}| < 1/4$.

4.) How can you boost the outcome of the randomized symmetric (unbiased) phase estimation, ensuring that you get a good estimate with exponentially high probability, while keeping the output distribution symmetric?

5.) What parameters should you choose for the Gaussian to ensure that the phase estimate produces an ε-precise outcome with probability at least $1-\delta$?

6.) (**H**) Let $|\psi_j\rangle\colon j \in [d]$ be an orthonormal eigenbasis of $U \in \mathbb{C}^{d\times d}$. Show that the phase estimation unitary V can be written in the following form:

$$V = \sum_{j\in[d]} |\psi_j\rangle\langle\psi_j| \otimes M(\varphi_j),$$

where the unitary matrix $M(\varphi_j)$ only depends on the eigenphase φ_j (but not on U or the actual eigenvector $|\psi_j\rangle$).

7.) Show that if you replace $H^{\otimes n}$ by QFT_N^{-1} in the phase estimation circuit, then the matrices $M(\varphi_j)$ in the decomposition of Exercise 6 become *shift invariant* in the sense that

$$M_{k,k+\ell}(\varphi_j) = M_{k',k'+\ell}(\varphi_j) \forall k,k',\ell \in \mathbb{Z}_N.$$

Hints

Exercise 3: Fix an eigenstate $|\psi\rangle$ and continuously change its eigenphase φ from 0 to 1/2 by appropriately changing U continuously. Arrive at a contradiction using a continuity argument.

Exercise 6: First prove that the subspaces of the form $|\psi_j\rangle \otimes \mathbb{C}^d$ are invariant under V. Then analyze the action of V within these subspaces.