Feel free to skip exercises that you find too easy or hard.

Exercises

1.) Finding marked elements using quantum walks. Given a symmetric (or reversible) Markov chain P, its largest eigenvalue λ_1 is always 1. Suppose its second largest (in absolute value) eigenvalue λ_2 satisfies $|\lambda_2| \leq 1-\delta$, and the probability that a vertex v is marked in the stationary distribution π is at least ε. It is known that under these conditions the hitting time of marked elements is at most $\frac{2}{\varepsilon\delta}$ [Gil14, Corollary 17]. Szegedy showed how to find a marked element in roughly the square root of this complexity using a quantum walk algorithm.

- As noted earlier, an intriguing property of Chebyshev polynomials is that [SV14]

$$x^t = \sum_{i=0}^{t} 2^{-t} \binom{t}{i} T_{|2i-t|}(x).$$

  By the Chernoff bound it follows that there is a parity-t degree $\mathcal{O}\left(\sqrt{t \log(1/\varepsilon)}\right)$-degree polynomial $p(x)$ such that $p(x) \in [-1, 1]$ and $|p(x) - x^t| \leq \varepsilon$ for all $x \in [-1, 1]$. Show that you can reduce all eigenvalues of P to less than ε via Quantum Eigenvalue Transformation with $\mathcal{O}\left(\sqrt{t \log(1/\varepsilon)}\right)$ uses of a block-encoding of P while keeping $\lambda_1 \geq 1 - \varepsilon$. As you will show in the homework it implies that it is possible to implement and ε-approximation of a block-encoding of $\Pi := |\sqrt{\pi}\rangle\langle\sqrt{\pi}|$ where $|\sqrt{\pi}\rangle$ with similar complexity.
- Given a block-encoding U_Π of the orthogonal projector $\Pi = (\langle 0^a| \otimes I) U_\Pi (|0^a\rangle \otimes I)$ implement the reflection operator $(2\Pi - I)$ with a few uses of U_Π and $U_\Pi^\dagger$.
- Give an algorithm inspired by Grover search that can find a marked element with $\mathcal{O}(1/\sqrt{\varepsilon})$ uses of $(2\Pi - I)$ starting from $\sqrt{\pi}$. Argue why it leads to an $\widetilde{\mathcal{O}}\left(\log(1/\varepsilon)/\sqrt{\varepsilon\delta}\right)$ algorithm for finding a marked element.

2.) Fixed-point amplitude amplification. Suppose A is a quantum circuit that prepares some n-qubit state $|\psi\rangle$, i.e., $A\colon |0^n\rangle \mapsto |\psi\rangle = \sqrt{1-p}|0\rangle|G\rangle + \sqrt{p}|1\rangle|B\rangle$, where $|G\rangle$ and $|B\rangle$ are some $(n-1)$-qubit pure states and $p \geq \theta$ for some known $\theta > 0$.

- You might use the fact that there is an odd polynomial $P(x)$ of degree $\mathcal{O}\left(\frac{1}{\theta}\log(1/\varepsilon)\right)$ such that $P(x) \geq 1 - \varepsilon$ for $x \in [\theta, 1]$ and $P(x) \in [-1, 1]$ for all $x \in [-1, 1]$.
- Give a quantum circuit U that acts as $A\colon |0^n\rangle \mapsto |G'\rangle$ where $\||G'\rangle - |0\rangle|G\rangle\| \leq \varepsilon$ regardless the value of $p \geq \theta$ and uses A and $A^\dagger$ only $\mathcal{O}\left(\frac{1}{\theta}\log(1/\varepsilon)\right)$ times.

3.) Moore-Penrose generalized (pseudo) inverse: for every $A \in \mathbb{C}^{n\times d}$ there is a unique matrix $A^+ \in \mathbb{C}^{d\times n}$ that satisfies the following 4 properties. $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^\dagger = (AA^+)$, and $(A^+A)^\dagger = (A^+A)$. (Therefore if A is invertible then $A^+ = A^{-1}$.)

- Show that if $A = \sum_{\sigma_i>0} \sigma_i |w_i\rangle\langle v_i|$ is a singular value decomposition, then $A^+ = \sum_{\sigma_i>0} \frac{1}{\sigma_i} |v_i\rangle\langle w_i|$.

References

[Gil14] András Gilyén. Quantum walk based search methods and algorithmic applications. Master's thesis, Eötvös Loránd University, 2014.

[SV14] Sushant Sachdeva and Nisheeth K. Vishnoi. Faster algorithms via approximation theory. *Found. Trends Theor. Comput. Sci.*, 9(2):125–210, 2014.