2023 Quantum Computing Homework Nr. 8

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Homework exercises – you can earn 10(+2) points in total! The second exercise is worth 2(+2) points while the others 4. On the last page you can find some hints where indicated by (**H**).

Reminder

Recall the *Spectral Theorem* (*Főtengely Tétel in Hungarian*): for every self-adjoint matrix $A \in \mathbb{C}^{d\times d}$ there exists a unitary $V \in \mathbb{C}^{d\times d}$ that diagonalizes A, i.e., $A = V \cdot D \cdot V^\dagger$ where $D \in \mathbb{R}^{d\times d}$ is a diagonal matrix containing the eigenvalues of A, and the columns of V are the corresponding eigenvectors. Let $|v_i\rangle$ be the i-th column of V and let λ_i be the i-th diagonal entry of D. In Dirac notation we can write this eigenvalue decomposition as $A = \sum_{i=0}^{d-1} \lambda_i |v_i\rangle\langle v_i|$.

Exercises

1.) A generalization of the Spectral Theorem for arbitrary matrices is called *Singular Value Decomposition.* This exercise walks you through a proof of this generalization by a reduction to the above spectral theorem. Let $A \in \mathbb{C}^{k\times d}$.

(a) Show that $A^\dagger A$ is self-adjoint and all of its eigenvalues are ≥ 0, so there is an eigenvalue decomposition $A^\dagger A = \sum_{i=0}^{d-1} \sigma_i^2 |v_i\rangle\langle v_i|$, where $\sigma_i \geq 0$ and the vectors $|v_i\rangle$ are orthonormal.

(b) Show that $\{|w_i\rangle := A|v_i\rangle/\sigma_i \colon \sigma_i > 0\}$ form an orthonormal system of vectors.

(c) Show that $A = \sum_{i\colon \sigma_i \neq 0} \sigma_i |w_i\rangle\langle v_i|$.

(d) Show that there exists unitaries $V \in \mathbb{C}^{d\times d}$, $W \in \mathbb{C}^{k\times k}$ and a diagonal matrix $\Sigma \in \mathbb{R}_{\geq 0}^{k\times d}$ containing the numbers σ_i on the diagonal such that $A = W \cdot \Sigma \cdot V^\dagger$.

The decomposition in Exercise 1c-1d is called Singular Value Decomposition. The numbers σ_i are called the singular values of A (usually ordered in decreasing order for convenience) and the vectors w_i and v_i are called the corresponding left and right singular vectors of A respectively.

2.) Linear combination of unitaries: Suppose that $A_0 = (\langle 0^a| \otimes I)U_0(|0^b\rangle \otimes I)$ and $A_1 = (\langle 0^a| \otimes I)U_1(|0^b\rangle \otimes I)$. Given $\alpha_0, \alpha_1 \in \mathbb{R}$ such that $|\alpha_0| + |\alpha_1| = 1$, construct a circuit that uses $U = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$ and two single qubit gates to implement a block-encoding V of $\alpha_0 A_0 + \alpha_1 A_1$ such that

$$(\langle 0^{a+1}| \otimes I)V(|0^{b+1}\rangle \otimes I) = \alpha_0 A_0 + \alpha_1 A_1. \qquad \textbf{(H)}$$

⋆ For 2 bonus points: construct a similar circuit in the general case $\alpha_0, \alpha_1 \in \mathbb{C}$.

3.) From Ronald's lecture notes [dW19, Chapter 9 Exercise 10]: Block-encoding an s-sparse Hermitian matrix A with $\|A\| \leq 1$ (see Section 9.4). Assume for simplicity that the entries of A are real.

(a) Show how to implement W_1 using an $O_{A,loc}$-query and a few other A-independent gates. For simplicity you may assume s is a power of 2 here, and you can use arbitrary single-qubit gates, possibly controlled by another qubit.
(Note that the same method allows to implement W_3.)

(b) Show how to implement W_2 using an O_A-query, an O_A^{-1}-query, and a few other A-independent gates (you may use auxiliary qubits as long as those start and end in $|0\rangle$). Note that W_2 just implements a rotation on the first qubit, by an angle that depends on A_{kj}. There's no need to write out circuits fully down to the gate-level here; it suffices if you describe the idea precisely.

(c) Show that the $(0^{n+1}i, 0^{n+1}j)$-entry of $W_3^{-1}W_1$ is $1/s$ if $A_{ij} \neq 0$, and is 0 if $A_{ij} = 0$.

(d) Show that the $(0^{n+1}i, 0^{n+1}j)$-entry of $W_3^{-1}W_2W_1$ is exactly A_{ij}/s.

Hints

Exercise 2: Remember from the lecture that $(\langle 0^{a+1}| \otimes I)(H \otimes I)U(H \otimes I)(|0^{b+1}\rangle \otimes I) = \frac{1}{2}(A_0 + A_1)$. Replace the Hadamrd gates with appropriate single-qubit gates.

References

[dW19] Ronald de Wolf. Quantum computing: Lecture notes, 2019. arXiv: `1907.09415v5`